Proposition 1. *The differential of S is*

$$dS = p\,dq - H\,dt,$$

hence S satisfies the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0.$$

Proof. Consider the integral curve $\gamma_{(q,t)}$ and the infinitesimal surface Σ obtained by varying the endpoint q to $q+dq$, hence the starting point from (q_0, p_0) to $(q_0, p+dp)$. The integral over Σ of $d\alpha$ vanishes. The proposition results from Stokes formula. □

References.

[1] V. Guillemin and S. Sternberg. *Symplectic techniques in physics.* Cambridge University Press, Cambridge, second edition, 1990.

3. Quasiperiodic motions

An important and simple class of Hamiltonians is that of *integrable* Hamiltonians, which do not depend on the angle θ. In such cases, the vector field becomes

$$\dot\theta = \frac{\partial H}{\partial r}(r) \equiv cst, \quad \dot r = 0,$$

and the flow

$$\varphi_t(\theta, r) = \left(\theta + t\frac{\partial H}{\partial r}(r), r\right).$$

The phase space is foliated in invariant tori $r = cst$, in restriction to which the flow is *quasiperiodic* (=linear), of *frequency vector* $\frac{\partial H}{\partial r}(r)$.

A vector r being fixed, let $\alpha := \frac{\partial H}{\partial r}(r) \in \mathbb{R}^n$ and consider the flow

$$\varphi_t : \mathbb{T}^n \to \mathbb{T}^n, \quad \theta \mapsto \theta + t\alpha.$$

Lemma 4. *The frequency vector α is a topological conjugacy invariant up to the action of the discrete group $GL_n(\mathbb{Z})$: if two linear flows $\theta + t\alpha$ and $\theta + t\beta$, with $\alpha, \beta \in \mathbb{R}^n$, are topologically conjugate, there exists $A \in GL_n(\mathbb{Z})$ such that $\beta = A\alpha$ (and, if the conjugacy preserves the orientation, $A \in SL_n(\mathbb{Z})$).*

Proof. Assume two linear flows $\theta+t\alpha$ and $\theta+t\beta$, with $\alpha, \beta \in \mathbb{R}^n$, are topologically conjugate: there exists a homeomorphism h of $\mathbb{T}^n$ such that $h(\theta+t\alpha) = h(\theta)+t\beta$. At the expense of substituting $h(\theta) - h(0)$ for $h(\theta)$, we may assume that $h(0) = 0$.

Let $H : \mathbb{R}^n \to \mathbb{R}^n$ be the unique lift of h such that $H(0) = 0$. Now, the equality $H(\theta + t\alpha) = H(\theta) + t\beta$ holds for $\theta = t = 0$ and, by continuity, for $\theta \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

Moreover, there exists a matrix $A \in GL_n(\mathbb{Z})$ such that $H(\theta+k) = H(\theta)+Ak$ for all $\theta \in \mathbb{R}^n$ and $k \in \mathbb{Z}^n$; A is invertible because H is. Hence $V := A^{-1}H - \mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ is a $\mathbb{Z}^n$-periodic vector field. In terms of V, the conjugacy hypothesis at $\theta = 0$ asserts that

$$L(t\alpha + V(t\alpha)) = LV(0) + t\beta \quad (\forall t \in \mathbb{R}),$$

i.e.

$$L(V(t\alpha) - V(0)) = t(\beta - L\alpha).$$

Since the left hand side is bounded, necessarily $\beta = L\alpha$. □

The action of $GL_n(\mathbb{Z})$ is closely related to the arithmetic properties of frequency vectors ; see [4, 2.2.3] for $n = 2$.

Proposition 2. *The following properties are equivalent :*

(1) *The vector α is* non resonant*: $k \cdot \alpha \neq 0$ for all $k \in \mathbb{Z}^n \setminus \{0\}$*
(2) *The flow (φ_t) of the constant vector field α is ergodic: invariant continuous functions ($f(\theta + t\alpha) \equiv f(\theta)$ for all $t \in \mathbb{R}$ and $\theta \in \mathbb{T}^n$) are constant*
(3) *For every continuous function f on $\mathbb{T}^n$, the time average of f exists, is constant and equals the space average of f:*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T f(\theta + t\alpha)\, dt = \int_{\mathbb{T}^n} f(\theta)\, d\theta.$$

(4) *Every trajectory of (φ_t) is dense on $\mathbb{T}^n$.*

More general classes of functions than continuous ones can be considered, but we lazily stick here to the most convenient setting for our purpose. See [1, 2, 3] for further results on ergodicity.

Proof. (1) ⇒ (2) Suppose that α is non resonant and let $f \in C^0(\mathbb{T}^1)$ be invariant: $f = f \circ \varphi_t$ for all t. The k-th Fourier coefficient of $f \circ \varphi_t$ is

$$\widehat{f \circ \varphi_t}(k) = \int_{\mathbb{T}^n} e^{-i2\pi k\cdot\theta} f(\theta + t\alpha)\, d\theta.$$

The change of variable $\theta' = \theta + t\alpha$ shows that

$$\widehat{f \circ \varphi_t}(k) = e^{i2\pi k\cdot\alpha t} \widehat{f}(k).$$

By uniqueness, for all $k \in \mathbb{Z}^n \setminus \{0\}$ we see that $\widehat{f}(k) = 0$. Hence f is constant.

(2) ⇒ (1) Conversely, suppose that $k \cdot \alpha = 0$ for some $k \in \mathbb{Z}^n \setminus \{0\}$. Then $f(\theta) = e^{i2\pi k\cdot\theta}$ is invariant and not constant, hence the flow is not ergodic.

(1) ⇒ (3) Call $\bar{f}$ the space-average of f. We will show the conclusion by taking more and more general functions.

– If f is constant, $\bar{f}(\theta) \equiv \bar{f}$ trivially. If $f(\theta) = e^{i2\pi k\cdot\theta}$ for some $k \in \mathbb{Z}^n \setminus \{0\}$, direct integration shows that

$$\frac{1}{T} \int_0^T f(\theta + t\alpha)\, d\theta = \frac{1}{T} e^{i2\pi k\cdot\theta} \frac{e^{i2\pi k\cdot\alpha T} - 1}{ik \cdot \alpha} \to_{T\to+\infty} 0 = \bar{f}.$$

The expression $k \cdot \alpha$ in the denominator is the first occurence of the so-called *small denominators*, which are the source of many difficulties in perturbation theory.

– If f is a trigonometric polynomial, the same conclusion holds by linearity.

– Let now f be continuous. Let $\epsilon > 0$. By the theorem of Weierstrass, there is a trigonometric polynomial P such that

$$\max_{\theta\in\mathbb{T}^n} |f(\theta) - P(\theta)| \leq \epsilon.$$

For such a P, we have shown that there is a time T_0 such that if $T \geq T_0$,

$$\left| \frac{1}{T} \int_0^T P(\theta + t\alpha)\, d\theta - \bar{P} \right| \leq \epsilon.$$

Using the two latter inequalities, we see that

$$\begin{aligned} &\left| \frac{1}{T} \int_0^T f(\theta + t\alpha)\, dt - \bar{f} \right| \\ &\leq \frac{1}{T} \int_0^T |f(\theta + t\alpha) - P(\theta + t\alpha)|\, dt + \left| \frac{1}{T} \int_0^T P(\theta + t\alpha)\, dt - \bar{P} \right| + |\bar{P} - \bar{f}| \leq 3\epsilon. \end{aligned}$$

So, again $\frac{1}{T} \int_0^T f(\theta + t\alpha)\, d\theta$ tends to 0.

(3) $\Rightarrow$ (1) Suppose α is resonant: $k \cdot \alpha = 0$ for some $k \in \mathbb{Z}^n \setminus \{0\}$, and let $f(\theta) = e^{i2\pi k\cdot\theta}$. The space average of f equals 0, while

$$\frac{1}{T} \int_0^T e^{i2\pi k\cdot(\theta+\alpha t)}\, dt = e^{i2\pi k\cdot\theta}.$$

So there exists a non constant continuous function whose time and space averages do not match.

(1) $\Rightarrow$ (4) Suppose that one trajectory is not dense: there exist a point $\theta \in \mathbb{T}^n$ and an open ball $B \subset \mathbb{T}^n$ such that the curve $t \mapsto \theta + t\alpha$ will never visit B. Let f be a continuous function whose support lies inside B and whose integral is > 0. The space average of f is > 0, while its time average is 0. Hence α is resonant.

(4) $\Rightarrow$ (1) Suppose α is resonant: $k \cdot \alpha = 0$ for some $k \in \mathbb{Z}^n \setminus \{0\}$. We will show that there is a small ball B centered at $\theta^o := k/2 \pmod{\mathbb{Z}^n}$ which the trajectory $t \mapsto t\alpha$ never visits. Indeed, let θ be in such a ball B of small radius. Does there exist $t \in \mathbb{R}$ such that $t\alpha = \theta$ in $\mathbb{T}^n$? Equivalently, does there exist $t \in \mathbb{R}$ and $\ell \in \mathbb{Z}^n$ such that $t\alpha = \theta + \ell$? Taking the dot product of the equation with k yields $0 = k \cdot \theta + k \cdot \ell$. But $k \cdot \ell \in \mathbb{Z}$, while $k \cdot \theta \in]0, 1[$ provided the radius of B is small enough (depending on k). This shows that there is no such $t \in \mathbb{R}$. $\square$

If we think for instance to two planets revloving around the Sun with frequencies α_1 and α_2, that the frequency vector $\alpha = (\alpha_1, \alpha_2)$ be resonant means that the two planets will regularly find themselves in the same relative position. Hence, their mutual attraction, which is small due to their small masses compared to the mass of the Sun, instead of averaging out, will pile up. This is all the more true that the order $|k| := |k_1| + \cdots + |k_n|$ of the resonance is small. As a general rule, perturbation theory rather studies what happens away from resonances, and at some distance away from them in the phase space (all the farther that they have low order).

References.

[1] V.I. Arnold and A. Avez. *Ergodic problems of classical mechanics.* Advanced book classics. Addison-Wesley, 1989.

[2] P.R. Halmos. *Lectures on ergodic theory.* AMS Chelsea Publishing Series. AMS Chelsea Pub., 2006.